Fast Toeplitz matrix-vector product (1)

As we know, the circulant matrix[e]

$$C_n = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \ddots & c_1 & c_0 \end{pmatrix} \tag{2}$$

can be diagonalized via FFTs, i.e.,

$$C_n = F_n^* \Lambda_n F_n, \;\; F_n(j,k) = \left[\frac{\omega^{(j-1)(k-1)}}{\sqrt{n}} \right]_{j,k=1,2,\ldots,n}, \;\; \omega = e^{-2\pi i/n},$$

where $i = \sqrt{-1}$ and F_n is the Fourier matrix.

[e] R.M. Gray, Toeplitz and Circulant Matrices: A Review, *Found. Trends Commun. Inf. Theory*, 2(3) (2005): 155-239.

Fast Toeplitz matrix-vector product (2)

To compute the matrix Λ_n, we use

$$\begin{aligned} C_n\mathbf{e}_1 &= (F_n^*\Lambda_n F_n)\mathbf{e}_1, \quad \mathbf{e}_1 = (1,0,\ldots,0)^T, \\ &\Rightarrow F_n(C_n\mathbf{e}_1) = \Lambda_n(F_n\mathbf{e}_1) = \Lambda_n\mathbf{1}, \\ &\Rightarrow \mathtt{fft}(\mathtt{C_n}(:,1)) = \mathtt{diag}(\Lambda_\mathtt{n}). \end{aligned}$$

For each Toeplitz[f] matrix T_n, we can embed it into a $2n \times 2n$ circulant matrix:

$$T_n\mathbf{v} \to C_{2n}\tilde{\mathbf{v}} = F_{2n}^*\Lambda_{2n}F_{2n}\tilde{\mathbf{v}} = \begin{bmatrix} T_n & \sharp \\ \sharp & T_n \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} T_n\mathbf{v} \\ \star \end{bmatrix}.$$

Refer to my GitHub codes 'toepmultip.m': https://github.com/Hsien-Ming-Ku/Group-of-FDEs/tree/master/Toeplitz_cir_precond

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[f] M. Ng, *Iterative Methods for Toeplitz Systems*, Oxford University Press, NY (2004).

Fast Toeplitz matrix-vector product (3)

To help you understand this process, we give you an example:

$$T_3 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 4 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3}$$

Then we have 6×6 circulant matrix:

$$T_3\mathbf{v} \to C_6\tilde{\mathbf{v}} = \begin{pmatrix} 1 & 2 & 3 & 0 & 5 & 4 \\ 4 & 1 & 2 & 3 & 0 & 5 \\ 5 & 4 & 1 & 2 & 3 & 0 \\ 0 & 5 & 4 & 1 & 2 & 3 \\ 3 & 0 & 5 & 4 & 1 & 2 \\ 2 & 3 & 0 & 5 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} T_3\mathbf{v} \\ \\ \star \\ \star \\ \star \end{pmatrix}, \tag{4}$$

where the zero elements can be replaced by other number.

Recruit MSc. students

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- Publication history: 34 SCI journal papers (see ResearchGate)
- Research topics: Matrix Computations, Krylov subspace solver-s, preconditioning, Numerical solutions of PDEs/FDEs;
- Funding: NSFC (No. 11801463, 01/01/2019 – 31/12/2021)
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