A low-rank Lie-Trotter splitting approach for nonlinear fractional complex Ginzburg-Landau equations

Yong-Liang Zhao

joint work with A. Ostermann (UIBK) and Xian-Ming Gu (SWUFE)

School of Mathematical Sciences,
University of Electronic Science and Technology of China,
Chengdu, Sichuan 611731, P.R. China

Chengdu • Sichuan, November 14-15, 2020

Outline

Outline

Background

Fractional Ginzburg-Landau equations (FGLEs) have been used to describe various physical phenomena such as neural networks modeling and fractal media[a].

[a] V.E. Tarasov and G.M. Zaslavsky. "Fractional Ginzburg-Landau equation for fractal media". In: Physica A 354 (2005), pp. 249–261.

[b] O. Koch and C. Lubich. "Dynamical low-rank approximation". In: SIAM J. Matrix Anal. Appl. 29 (2007), pp. 434–454.

Background

Fractional Ginzburg-Landau equations (FGLEs) have been used to describe various physical phenomena such as neural networks modeling and fractal media[a].

Dynamical low-rank approximations of matrices are widely used for reducing models of large size. Such an approach has a broad variety of application areas, such as image compression, information retrieval and a blow-up problem of a reaction-diffusion equation[b].

[a] V.E. Tarasov and G.M. Zaslavsky. "Fractional Ginzburg-Landau equation for fractal media". In: Physica A 354 (2005), pp. 249–261.

[b] O. Koch and C. Lubich. "Dynamical low-rank approximation". In: SIAM J. Matrix Anal. Appl. 29 (2007), pp. 434–454.

Outline

Motivation

In this work, we mainly study a dynamical low-rank approximation for solving the following 2D complex FGLE:

$$\begin{cases} \partial_t u - (\nu + \mathbf{i}\eta)(\partial_x^\alpha + \partial_y^\beta)u + (\kappa + \mathbf{i}\xi)|u|^2 u - \gamma u = 0, \\ \qquad\qquad\qquad\qquad (x, y, t) \in \Omega \times (0, T], \\ u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \\ u(x, y, t) = 0, \qquad (x, y) \in \partial\Omega, \end{cases} \tag{1}$$

- $\mathbf{i} = \sqrt{-1}$, $\nu > 0$, $\kappa > 0$, η, ξ, γ are real numbers;
- $\Omega = (x_L, x_R) \times (y_L, y_R) \subset \mathbb{R}^2$, $u_0(x, y)$ is a given complex function.

Motivation

where both ∂_x^α $(1 < \alpha < 2)$ and ∂_y^β $(1 < \beta < 2)$ are the Riesz fractional derivatives[c]:

$$\partial_x^\alpha u(x, y, t) = -\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} |x - \zeta|^{1-\alpha} u(\zeta, y, t) d\zeta,$$

$$\partial_y^\beta u(x, y, t) = -\frac{1}{2\cos(\beta\pi/2)\Gamma(2-\beta)} \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} |y - \zeta|^{1-\beta} u(x, \zeta, t) d\zeta.$$

[c] R. Gorenflo and F. Mainardi. "Random walk models for space-fractional diffusion processes". In: Fract. Calc. Appl. Anal. 1 (1998), pp. 167–191.

Outline

The semi-discrete scheme

The second-order fractional centered difference method[d] is used for space discretization, i.e,

$$\partial_x^\alpha u(x_i, y_j, t) = -h_x^{-\alpha} \sum_{k=-N_x+i}^{i} g_k^\alpha u_{i-k,j}(t) + \mathcal{O}(h_x^2)$$
$$= \delta_x^\alpha u_{ij}(t) + \mathcal{O}(h_x^2)$$

$$\partial_y^\beta u(x_i, y_j, t) = -h_y^{-\beta} \sum_{k=-N_y+j}^{j} g_k^\beta u_{i,j-k}(t) + \mathcal{O}(h_y^2)$$
$$= \delta_y^\beta u_{ij}(t) + \mathcal{O}(h_y^2)$$

[d] C. Çelik and M. Duman. "Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative". In: J. Comput. Phys. 231 (2012), pp. 1743–1750.

The semi-discrete scheme

$$g_k^{\mu} = \frac{(-1)^k \Gamma(1+\mu)}{\Gamma(\mu/2 - k + 1)\Gamma(\mu/2 + k + 1)} \quad (\mu = \alpha,\ \beta,\ k \in \mathbb{Z})$$

$$h_x = \frac{x_R - x_L}{N_x}, \quad h_y = \frac{y_R - y_L}{N_y}$$

The semi-discrete scheme

$$g_k^{\mu} = \frac{(-1)^k \Gamma(1+\mu)}{\Gamma(\mu/2 - k + 1)\Gamma(\mu/2 + k + 1)} \quad (\mu = \alpha,\ \beta,\ k \in \mathbb{Z})$$

$$h_x = \frac{x_R - x_L}{N_x}, \quad h_y = \frac{y_R - y_L}{N_y}$$

The semi-discrete scheme is given as

$$\frac{du_{ij}(t)}{dt} = (\nu + \mathbf{i}\eta)\left(\delta_x^{\alpha} + \delta_y^{\beta}\right) u_{ij}(t) - (\kappa + \mathbf{i}\xi)\left|u_{ij}(t)\right|^2 u_{ij}(t) + \gamma u_{ij}(t),$$

where $u_{ij}(t) \approx u(x_i, y_j, t)$.

The matrix differential equation

The matrix differential equation corresponding to the above spatial semi-discretized form is give by

$$\begin{cases} \dot{U}(t) = A_x U(t) + U(t) A_y - (\kappa + \mathbf{i}\xi) |U(t)|^2 U(t) + \gamma U(t), \\ U(0) = U^0, \end{cases} \tag{2}$$

where

$$U(t) = [u_{ij}(t)]_{\substack{1 \le i \le N_x - 1 \\ 1 \le j \le N_y - 1}}, \quad U^0 = [u_0(x_i, y_j)]_{\substack{1 \le i \le N_x - 1 \\ 1 \le j \le N_y - 1}},$$

$$\dot{U}(t) = \left[\frac{du_{ij}(t)}{dt} \right]_{\substack{1 \le i \le N_x - 1 \\ 1 \le j \le N_y - 1}},$$

A_x and A_y are two symmetric Toeplitz matrices with first columns:

$$-\frac{\nu + \mathbf{i}\eta}{h_x^\alpha} \left[g_0^\alpha, g_1^\alpha, \cdots, g_{N_x-2}^\alpha \right]^T \quad \text{and} \quad -\frac{\nu + \mathbf{i}\eta}{h_y^\beta} \left[g_0^\beta, g_1^\beta, \cdots, g_{N_y-2}^\beta \right]^T.$$

Outline

The full-rank Lie-Trotter splitting method

Eq. (2) is split into the following two subproblems:

$$\dot{U}_1(t) = \overbrace{A_x U_1(t) + U_1(t) A_y}^{\text{Stiff linear part}}, \quad U_1(t_0) = U_1^0, \tag{3}$$

and

$$\begin{cases} \dot{U}_2(t) = G(U_2(t)) \triangleq \underbrace{-(\kappa + \mathbf{i}\xi) \left|U_2(t)\right|^2 U_2(t) + \gamma U_2(t)}_{\text{Nonstiff (nonlinear) part}}, \\ U_2(t_0) = U_2^0. \end{cases} \tag{4}$$

The full-rank Lie-Trotter splitting method

The full-rank Lie-Trotter splitting scheme with time step size $\tau = \frac{T}{M}$ is given by

$$\mathcal{L}_\tau = \Phi_\tau^L \circ \Phi_\tau^G.$$

Here, Φ_τ^L and Φ_τ^G denote the solutions of Eqs. (3) and (4), respectively.

The full-rank Lie-Trotter splitting method

The full-rank Lie-Trotter splitting scheme with time step size $\tau = \frac{T}{M}$ is given by

$$\mathcal{L}_\tau = \Phi_\tau^L \circ \Phi_\tau^G.$$

Here, Φ_τ^L and Φ_τ^G denote the solutions of Eqs. (3) and (4), respectively.

Starting with $U_2^0 = U^0$, the numerical solution U^1 of Eq. (1) at $t = t_1$ is thus given by

$$U^1 = \mathcal{L}_\tau(U^0) = \Phi_\tau^L \circ \Phi_\tau^G(U^0).$$

Subsequently, the numerical solution of Eq. (1) at t_k is

$$U^k = \mathcal{L}_\tau^k(U^0).$$

Outline

The low-rank approximation

We seek after low-rank approximations

$$X_1(t), X_2(t) \in \mathcal{M}_r = \left\{ X(t) \in \mathbb{C}^{(N_x-1)\times(N_y-1)} \mid \mathrm{rank}\left(X(t)\right) = r \right\}$$

to $U_1(t)$ and $U_2(t)$, respectively.

The low-rank solution of Eq. (3)

It can be observed that (3) is rank preserving[e]. That is, for any $X \in \mathcal{M}_r$, $A_x X + X A_y \in \mathcal{T}_X \mathcal{M}_r$, where $\mathcal{T}_X \mathcal{M}_r$ is the tangent space of $\mathcal{M}_r$ at a rank-r matrix X.

[e] U. Helmke and J. B. Moore. Optimization and Dynamical Systems. London: Springer-Verlag, 1994, Lemma 1.22.

The low-rank solution of Eq. (3)

It can be observed that (3) is rank preserving[e]. That is, for any $X \in \mathcal{M}_r$, $A_x X + X A_y \in \mathcal{T}_X \mathcal{M}_r$, where $\mathcal{T}_X \mathcal{M}_r$ is the tangent space of $\mathcal{M}_r$ at a rank-r matrix X. Thus, for a given rank-r initial value X_1^0, the solution of

$$\dot{X}_1(t) = A_x X_1(t) + X_1(t) A_y, \quad X_1(t_0) = X_1^0$$

remains rank-r for all t.

How to solve this at $t = t_1$?

[e] U. Helmke and J. B. Moore. Optimization and Dynamical Systems. London: Springer-Verlag, 1994, Lemma 1.22.

The low-rank solution of Eq. (3)

It can be observed that (3) is rank preserving[e]. That is, for any $X \in \mathcal{M}_r$, $A_x X + X A_y \in \mathcal{T}_X \mathcal{M}_r$, where $\mathcal{T}_X \mathcal{M}_r$ is the tangent space of $\mathcal{M}_r$ at a rank-r matrix X. Thus, for a given rank-r initial value X_1^0, the solution of

$$\dot{X}_1(t) = A_x X_1(t) + X_1(t) A_y, \quad X_1(t_0) = X_1^0$$

remains rank-r for all t.

How to solve this at $t = t_1$?

$$X_1(t_1) = e^{\tau A_x} X_1^0 e^{\tau A_y}.$$

[e] U. Helmke and J. B. Moore. Optimization and Dynamical Systems. London: Springer-Verlag, 1994, Lemma 1.22.

Thinking in terms of the low-rank manifold $\mathcal{M}$ and the orthogonal projection onto the tangent space $\mathcal{T}_{\mathbf{Y}(t)}\mathcal{M}$, we imagine condition (1.6) as

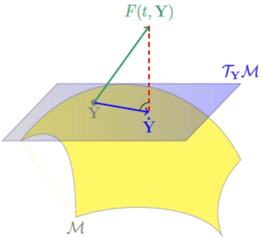

Figure 1.1: Orthogonal projection onto the tangent space of the low-rank manifold. The red dashed line represents the orthogonal projection, which results in $\dot{\mathbf{Y}}$. Out of all $\delta\mathbf{Y} \in \mathcal{T}_{\mathbf{Y}}\mathcal{M}$, $\dot{\mathbf{Y}}$ is the tangent element that minimizes the distance between $F(t, \mathbf{Y})$ and the tangent space of $\mathcal{M}$ at the approximation matrix $\mathbf{Y}$.

H. M. Walach,Time integration for the dynamical low-rank approximation of matrices and tensors (Doctoral dissertation), Eberhard Karls Universität Tübingen (2019), Page 17.

The low-rank solution of Eq. (4)

The low-rank solution of subproblem (4) is obtained by solving the following optimization problem[f]

$$\min_{X_2(t)\in\mathcal{M}_r} \left\|\dot{X}_2(t) - \dot{U}_2(t)\right\|, \quad \text{s.t. } \dot{X}_2(t) \in \mathcal{T}_{X_2(t)}\mathcal{M}_r,$$

where $\mathcal{T}_{X_2(t)}\mathcal{M}_r$ is the tangent space of $\mathcal{M}_r$ at the current approximation $X_2(t)$.

How to solve this optimization problem?

[f] O. Koch and C. Lubich. "Dynamical low-rank approximation". In: SIAM J. Matrix Anal. Appl. 29 (2007), pp. 434–454.

The low-rank solution of Eq. (4)

The low-rank solution of subproblem (4) is obtained by solving the following optimization problem[f]

$$\min_{X_2(t)\in\mathcal{M}_r} \left\|\dot{X}_2(t) - \dot{U}_2(t)\right\|, \quad \text{s.t. } \dot{X}_2(t) \in \mathcal{T}_{X_2(t)}\mathcal{M}_r,$$

where $\mathcal{T}_{X_2(t)}\mathcal{M}_r$ is the tangent space of $\mathcal{M}_r$ at the current approximation $X_2(t)$.

How to solve this optimization problem?

C. Lubich, I. V. Oseledets, A projector-splitting integrator for dynamical low-rank approximation, BIT 54 (2014) 171-188.

[f] O. Koch and C. Lubich. "Dynamical low-rank approximation". In: SIAM J. Matrix Anal. Appl. 29 (2007), pp. 434–454.

Solve the optimization problem

$$\min_{X_2(t)\in\mathcal{M}_r} \left\| \dot{X}_2(t) - \dot{U}_2(t) \right\|, \quad \text{s.t. } \dot{X}_2(t) \in \mathcal{T}_{X_2(t)}\mathcal{M}_r$$

equivalent to

$$\dot{X}_2(t) = P(X_2(t))G(X_2(t)), \quad X_2(t_0) = X_2^0 \in \mathcal{M}_r,$$

where $P(X_2(t))$ is the orthogonal projection onto $\mathcal{T}_{X_2(t)}\mathcal{M}_r$.

Solve the optimization problem

A rank-r matrix $X_2(t) \in \mathbb{C}^{(N_x-1)\times(N_y-1)}$ can be expressed as $X_2(t) = S(t)\Sigma(t)V(t)^*$, where $S(t) \in \mathbb{C}^{(N_x-1)\times r}$ and $V(t) \in \mathbb{C}^{(N_y-1)\times r}$ have orthonormal columns, $\Sigma(t) \in \mathbb{C}^{r\times r}$ is nonsingular and has the same singular values as $X_2(t)$, and * means conjugate transpose[g].

This expression is similar to SVD, but $\Sigma(t)$ is not necessarily a diagonal matrix.

[g] C. Lubich and I. V. Oseledets. "A projector-splitting integrator for dynamical low-rank approximation". In: BIT 54 (2014), pp. 171–188.

The projector-splitting integrator

$$
\begin{aligned}
P(X_2(t))G(X_2(t)) = {} & S(t)S(t)^* G(X_2(t)) - \\
& S(t)S(t)^* G(X_2(t))V(t)V(t)^* + \\
& G(X_2(t))V(t)V(t)^* \\
\triangleq {} & P_1(X_2(t))G(X_2(t)) - \\
& P_1(X_2(t))G(X_2(t))P_2(X_2(t)) + \\
& G(X_2(t))P_2(X_2(t))
\end{aligned}
$$

$P_1(X_2(t))$ and $P_2(X_2(t))$ are the orthogonal projections onto the spaces spanned by the range and the corange of $X_2(t)$, respectively.

The projector-splitting integrator

The low-rank solution of Eq. (4) at t_1 can be obtained by solving the evolution equations:

$$\dot{X}_2^{I}(t) = P_1(X_2(t))G(X_2(t)), \quad X_2^{I}(t_0) = X_2^0,$$
$$\dot{X}_2^{II}(t) = -P_1(X_2(t))G(X_2(t))P_2(X_2(t)), \quad X_2^{II}(t_0) = X_2^{I}(t_1),$$
$$\dot{X}_2^{III}(t) = G(X_2(t))P_2(X_2(t)), \quad X_2^{III}(t_0) = X_2^{II}(t_1).$$

Then, $X_2^{III}(t_1)$ is the approximate solution of $X_2(t_1)$.

The low-rank approximation of Eq. (1)

Let X^0 be a rank-r approximation of the initial value U^0. We start with $X_2^0 = X^0$ and obtain the rank-r approximation X^1 of the solution of (1) at t_1 as

$$X^1 = \mathcal{L}_{\tau,r}(X^0) = \Phi_\tau^L \circ \tilde{\Phi}_\tau^G(X^0). \tag{5}$$

Here, $\tilde{\Phi}_\tau^G$ denotes the low-rank solution of (4). Consequently, the low-rank solution of (1) at t_k is $X^k = \mathcal{L}_{\tau,r}^k(X^0)$.

Outline

Preliminaries

Assumption 1

We assume that

(a) *G is continuously differentiable in a neighborhood of the exact solution, and the solution of Eq. (1) is bounded, i.e.* $|u(x,y,t)| \leq \delta$, $(x,y,t) \in \Omega \times (0,T]$, *for some* $\delta > 0$*;*

(b) *there exists* $\varepsilon > 0$ *such that*

$$G(X(t)) = \tilde{B}(X(t)) + R(X(t)) \quad \mathit{for} \quad t_0 \leq t \leq T,$$

where $\tilde{B}(X(t)) \in \mathcal{T}_{X(t)}\mathcal{M}_r$ *and* $\|R(X(t))\|_F \leq \varepsilon$.

(c) *The exact solution of (1) is sufficiently smooth such that the fractional central difference method is second-order accurate.*

Preliminaries

Property 1

(a) *There exists $C_1 > 0$ such that A_x and A_y satisfy*

$$\left\| e^{tA_x} Z e^{tA_y} \right\|_F \leq \|Z\|_F \,, \quad \left\| e^{tA_x} (A_x Z + Z A_y) e^{tA_y} \right\|_F \leq \frac{C_1}{t} \|Z\|_F$$

for all $t > 0$ and $Z \in \mathbb{C}^{(N_x-1)\times(N_y-1)}$.

(b) *Under Assumption 1(a), the function G is locally Lipschitz continuous and bounded in a neighborhood of the solution $U(t)$. That is to say, for $\left\| \hat{U} - U(t) \right\|_F \leq \tilde{\xi}$, $\left\| \tilde{U} - U(t) \right\|_F \leq \tilde{\xi}$ and $\left\| \bar{U} - U(t) \right\|_F \leq \tilde{\xi}$ ($\tilde{\xi} > 0$, $t_0 \leq t \leq T$), one obtains*

$$\left\| G(\hat{U}) - G(\tilde{U}) \right\|_F \leq L \left\| \hat{U} - \tilde{U} \right\|_F , \quad \left\| G(\bar{U}) \right\|_F \leq H,$$

where the constants L and H depend on δ and $\tilde{\xi}$.

Outline

Convergence analysis

Let X^0 be a given rank-r approximation of the initial value U^0 satisfying $\left\|X^0 - U^0\right\|_F \leq \sigma$, for some $\sigma \geq 0$.

Convergence analysis

The global error can be split in three terms:

1) The global error of the full-rank Lie-Trotter splitting, i.e.

$$E_{fs}^k = \mathcal{U}(t_k) - \mathcal{L}_\tau^k(U^0).$$

Here, $\mathcal{U}(t_k) = [u(x_i, y_j, t_k)]_{1 \le i,j \le N-1}$.

Convergence analysis

The global error can be split in three terms:

1) The global error of the full-rank Lie-Trotter splitting, i.e.

$$E_{fs}^k = \mathcal{U}(t_k) - \mathcal{L}_\tau^k(U^0).$$

Here, $\mathcal{U}(t_k) = [u(x_i, y_j, t_k)]_{1\leq i,j\leq N-1}$.

2) The difference between the full-rank initial value U^0 and its rank-r approximation X^0, both propagated by the full-rank Lie-Trotter splitting, i.e.

$$E_{fl}^k = \mathcal{L}_\tau^k(U^0) - \mathcal{L}_\tau^k(X^0).$$

Convergence analysis

The global error can be split in three terms:

1) The global error of the full-rank Lie-Trotter splitting, i.e.

$$E_{fs}^k = \mathcal{U}(t_k) - \mathcal{L}_\tau^k(U^0).$$

Here, $\mathcal{U}(t_k) = [u(x_i, y_j, t_k)]_{1 \le i,j \le N-1}$.

2) The difference between the full-rank initial value U^0 and its rank-r approximation X^0, both propagated by the full-rank Lie-Trotter splitting, i.e.

$$E_{fl}^k = \mathcal{L}_\tau^k(U^0) - \mathcal{L}_\tau^k(X^0).$$

3) The difference between the full-rank Lie-Trotter splitting and the low-rank splitting applied to X^0

$$E_{lr}^k = \mathcal{L}_\tau^k(X^0) - \mathcal{L}_{\tau,r}^k(X^0).$$

Convergence analysis

Theorem 1

Under Assumption 1, for $1 \leq k \leq M$, the error bound

$$\|E_{fs}^k\|_F \leq C_2[\tau(1+|\log\tau|)+h_x^2+h_y^2]$$

holds. Here, the constant C_2 depends on C_1, L and H.

Convergence analysis

Theorem 1

Under Assumption 1, for $1 \le k \le M$, the error bound

$$\|E_{fs}^k\|_F \le C_2[\tau(1 + |\log \tau|) + h_x^2 + h_y^2]$$

holds. Here, the constant C_2 depends on C_1, L and H.

Hints: $\|E_{fs}^k\|_F = \|\mathcal{U}(t_k) - U(t_k) + U(t_k) - \mathcal{L}_\tau^k(U^0)\|_F$ and Property 1 is used.

Convergence analysis

Theorem 2

Under Assumption 1, E_{lr}^k is bounded on $t_0 \leq t_0 + k\tau \leq T$ as

$$\|E_{lr}^k\|_F \leq C_3\varepsilon + C_4\tau,$$

where the constants C_3 and C_4 depend on H, L and T.

Convergence analysis

Theorem 2

Under Assumption 1, E_{lr}^k is bounded on $t_0 \leq t_0 + k\tau \leq T$ as

$$\|E_{lr}^k\|_F \leq C_3\varepsilon + C_4\tau,$$

where the constants C_3 and C_4 depend on H, L and T.

Hints: A. Ostermann, C. Piazzola, H. Walach, Convergence of a low-rank Lie-Trotter splitting for stiff matrix differential equations, SIAM J. Numer. Anal., 57 (2019) 1947-1966.

Convergence analysis

Theorem 3

Under Assumption 1, there exists $\tilde{\tau}$ such that for all $0 < \tau \leq \tilde{\tau}$, the error of $\mathcal{L}_{\tau,r}$ is bounded on $t_0 \leq t_0 + k\tau \leq T$ by

$$\left\|\mathcal{U}(t_k) - \mathcal{L}_{\tau,r}^{k}(X^0)\right\|_F \leq C_3\varepsilon + C_5[\tau(1+|\log\tau|) + h_x^2 + h_y^2] + e^{LT}\sigma. \tag{6}$$

Here C_3 and C_5 (containing C_2 and C_4) are independent of τ and k.

Convergence analysis

Theorem 3

Under Assumption 1, there exists $\tilde{\tau}$ such that for all $0 < \tau \leq \tilde{\tau}$, the error of $\mathcal{L}_{\tau,r}$ is bounded on $t_0 \leq t_0 + k\tau \leq T$ by

$$\left\| \mathcal{U}(t_k) - \mathcal{L}_{\tau,r}^k(X^0) \right\|_F \leq C_3\varepsilon + C_5[\tau(1 + |\log\tau|) + h_x^2 + h_y^2] + e^{LT}\sigma. \quad (6)$$

Here C_3 and C_5 (containing C_2 and C_4) are independent of τ and k.

Hints: $\left\| \mathcal{U}(t_k) - \mathcal{L}_{\tau,r}^k(X^0) \right\|_F = \left\| E_{fs}^k + E_{fl}^k + E_{lr}^k \right\|_F$, $\|E_{fl}^k\|_F \leq e^{LT}\sigma$.

Convergence analysis

Theorem 3

Under Assumption 1, there exists $\tilde{\tau}$ such that for all $0 < \tau \leq \tilde{\tau}$, the error of $\mathcal{L}_{\tau,r}$ is bounded on $t_0 \leq t_0 + k\tau \leq T$ by

$$\left\|\mathcal{U}(t_k) - \mathcal{L}^k_{\tau,r}(X^0)\right\|_F \leq C_3\varepsilon + C_5[\tau(1 + |\log\tau|) + h_x^2 + h_y^2] + e^{LT}\sigma. \tag{6}$$

Here C_3 and C_5 (containing C_2 and C_4) are independent of τ and k.

Hints: $\|\mathcal{U}(t_k) - \mathcal{L}^k_{\tau,r}(X^0)\|_F = \|E^k_{fs} + E^k_{fl} + E^k_{lr}\|_F$, $\|E^k_{fl}\|_F \leq e^{LT}\sigma$.

The spatial error does not depend on r, but when r is small, the error is dominated by the term $C_3\varepsilon$.

Numerical results

Example X Considering the equation (1) with $\Omega = [-10, 10] \times [-10, 10]$ with $u_0(x, y) = 2\,\mathrm{sech}(x)\,\mathrm{sech}(y) e^{3\mathrm{i}(x+y)}$, $\nu = \eta = \kappa = \xi = \gamma = 1$ and $T = 1$.

We fix $N_x = N_y = N$ and $h_x = h_y = h$. The reference solution $((N, M) = (512, 10000))$ is computed by the LBDF2 scheme[h]. Let

$$\mathrm{relerr}(\tau, h) = \frac{\left\| X^M - \mathcal{U}(T) \right\|_F}{\left\| \mathcal{U}(T) \right\|_F}.$$

[h] Q. Zhang et al. "Linearized ADI schemes for two-dimensional space-fractional nonlinear Ginzburg–Landau equation". In: Comput. Math. Appl. 80 (2020), pp. 1201–1220.

Numerical results

Table: Errors and observed temporal convergence orders for $N = 512$ for Example X.

(α, β)	M	$r=1$ relerr(τ, h)	$r=1$ rate$_\tau$	$r=2$ relerr(τ, h)	$r=2$ rate$_\tau$	$r=3$ relerr(τ, h)	$r=3$ rate$_\tau$
(1.2, 1.9)	16	4.8609E-01	–	8.2522E-01	–	7.9346E-01	–
	64	5.3468E-01	-0.0687	2.0577E-01	1.0019	2.0557E-01	0.9743
	256	5.7709E-01	-0.0551	5.2778E-02	0.9815	5.1313E-02	1.0011
	1024	5.8871E-01	-0.0144	2.4422E-02	0.5559	1.3848E-02	0.9448
(1.5, 1.5)	16	5.5049E-01	–	7.9464E-01	–	7.6149E-01	–
	64	6.4759E-01	-0.1172	1.9941E-01	0.9973	1.9988E-01	0.9648
	256	6.9229E-01	-0.0481	5.4117E-02	0.9408	5.0664E-02	0.9901
	1024	7.0413E-01	-0.0122	3.0728E-02	0.4083	1.4516E-02	0.9017
(1.7, 1.3)	16	5.2921E-01	–	7.8605E-01	–	7.5265E-01	–
	64	6.0475E-01	-0.0962	1.9632E-01	1.0007	1.9643E-01	0.9690
	256	6.4751E-01	-0.0493	5.2519E-02	0.9511	4.9500E-02	0.9943
	1024	6.5899E-01	-0.0127	2.8515E-02	0.4406	1.3823E-02	0.9202
(1.9, 1.2)	16	4.9746E-01	–	8.3250E-01	–	7.9540E-01	–
	64	5.3481E-01	-0.0522	2.0620E-01	1.0067	2.0586E-01	0.9750
	256	5.7706E-01	-0.0548	5.2760E-02	0.9833	5.1371E-02	1.0013
	1024	5.8870E-01	-0.0144	2.4418E-02	0.5557	1.3867E-02	0.9446

Numerical results

Table: Errors and observed temporal convergence orders for $N = 512$ for Example X.

(α, β)	M	$r = 4$ relerr(τ, h)	$r = 4$ rate$_\tau$	$r = 5$ relerr(τ, h)	$r = 5$ rate$_\tau$
(1.2, 1.9)	16	7.9240E-01	–	7.9253E-01	–
	64	2.0455E-01	0.9769	2.0499E-01	0.9755
	256	5.0386E-02	1.0107	5.0491E-02	1.0107
	1024	1.2499E-02	1.0056	1.2581E-02	1.0024
(1.5, 1.5)	16	7.5911E-01	–	7.5953E-01	–
	64	1.9839E-01	0.9680	1.9883E-01	0.9668
	256	4.9256E-02	1.0050	4.9399E-02	1.0045
	1024	1.2248E-02	1.0039	1.2361E-02	0.9993
(1.7, 1.3)	16	7.5042E-01	–	7.5024E-01	–
	64	1.9513E-01	0.9716	1.9547E-01	0.9702
	256	4.8334E-02	1.0067	4.8461E-02	1.0060
	1024	1.2008E-02	1.0045	1.2113E-02	1.0001
(1.9, 1.2)	16	7.9353E-01	–	7.9298E-01	–
	64	2.0476E-01	0.9772	2.0499E-01	0.9759
	256	5.0393E-02	1.0113	5.0492E-02	1.0107
	1024	1.2499E-02	1.0057	1.2581E-02	1.0024

Numerical results

Table: Errors and observed spatial convergence orders for $M = 10000$ for Example X.

(α, β)	N	$r=1$ relerr(τ, h)	$r=1$ rate$_h$	$r=2$ relerr(τ, h)	$r=2$ rate$_h$	$r=3$ relerr(τ, h)	$r=3$ rate$_h$
(1.2, 1.9)	32	7.1496E-01	–	6.0075E-01	–	6.0033E-01	–
	64	5.8194E-01	0.2970	1.2508E-01	2.2639	1.2512E-01	2.2624
	128	5.8821E-01	-0.0155	3.5052E-02	1.8353	2.9685E-02	2.0755
	256	5.9137E-01	-0.0077	2.2788E-02	0.6212	7.9685E-03	1.8974
(1.5, 1.5)	32	8.2826E-01	–	6.2358E-01	–	6.2338E-01	–
	64	6.9735E-01	0.2482	1.2391E-01	2.3313	1.2393E-01	2.3306
	128	7.0396E-01	-0.0136	3.8797E-02	1.6753	2.9802E-02	2.0560
	256	7.0691E-01	-0.0060	2.9772E-02	0.3820	9.2925E-03	1.6813
(1.7, 1.3)	32	7.9484E-01	–	6.2588E-01	–	6.2572E-01	–
	64	6.5274E-01	0.2842	1.2632E-01	2.3088	1.2627E-01	2.3090
	128	6.5868E-01	-0.0131	3.7824E-02	1.7397	3.0045E-02	2.0713
	256	6.6165E-01	-0.0065	2.7462E-02	0.4619	8.6706E-03	1.7929
(1.9, 1.2)	32	7.1495E-01	–	6.0075E-01	–	6.0033E-01	–
	64	5.8194E-01	0.2970	1.2507E-01	2.2640	1.2512E-01	2.2624
	128	5.8821E-01	-0.0155	3.5052E-02	1.8352	2.9686E-02	2.0755
	256	5.9137E-01	-0.0077	2.2788E-02	0.6212	7.9705E-03	1.8970

Numerical results

Table: Errors and observed spatial convergence orders for $M = 10000$ for Example X.

(α, β)	N	$r = 4$ relerr(τ, h)	$r = 4$ rate$_h$	$r = 5$ relerr(τ, h)	$r = 5$ rate$_h$
(1.2, 1.9)	32	6.0079E-01	–	6.0069E-01	–
	64	1.2478E-01	2.2675	1.2482E-01	2.2668
	128	2.8836E-02	2.1134	2.8893E-02	2.1111
	256	6.1959E-03	2.2185	6.2558E-03	2.2075
(1.5, 1.5)	32	6.2391E-01	–	6.2380E-01	–
	64	1.2330E-01	2.3392	1.2337E-01	2.3381
	128	2.8319E-02	2.1223	2.8406E-02	2.1187
	256	6.1664E-03	2.1993	6.2448E-03	2.1855
(1.7, 1.3)	32	6.2618E-01	–	6.2608E-01	–
	64	1.2579E-01	2.3156	1.2585E-01	2.3146
	128	2.8901E-02	2.1218	2.8976E-02	2.1188
	256	6.2136E-03	2.2176	6.2862E-03	2.2046
(1.9, 1.2)	32	6.0079E-01	–	6.0069E-01	–
	64	1.2478E-01	2.2675	1.2482E-01	2.2668
	128	2.8836E-02	2.1134	2.8893E-02	2.1111
	256	6.1959E-03	2.2185	6.2558E-03	2.2075

Numerical results

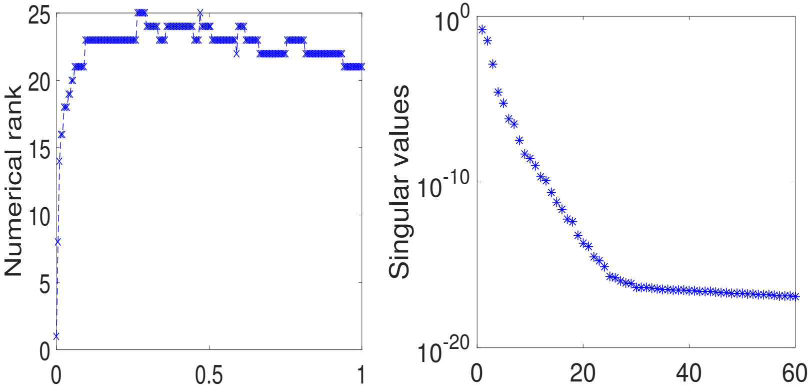

Figure: Results for Example X for $(\alpha, \beta) = (1.5, 1.5)$ and $N = M = 200$. Left: Numerical rank of the LBDF2 solution as a function of t. Right: First 60 singular values of the LBDF2 solution at $t = T$.

Numerical results

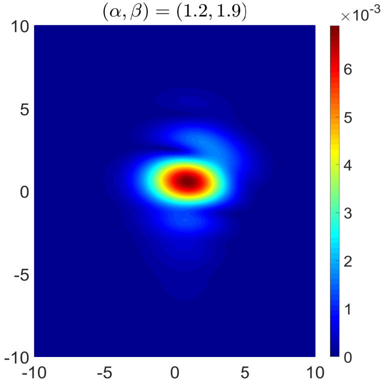

Figure: Comparison of the absolute values of the LBDF2 solution and our low-rank solution at $t = T$ for $(N, M) = (512, 200)$ and $(\alpha, \beta) = (1.2, 1.9)$ for Example X. Left: The LBDF2 solution. Right: The low-rank solution (rank $r = 5$).

Numerical results

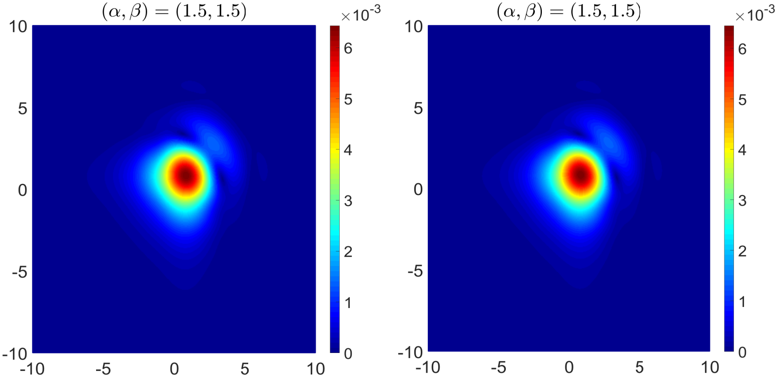

Figure: Comparison of the absolute values of the LBDF2 solution and our low-rank solution at $t = T$ for $(N, M) = (512, 200)$ and $(\alpha, \beta) = (1.5, 1.5)$ for Example X. Left: The LBDF2 solution. Right: The low-rank solution (rank $r = 5$).

Numerical results

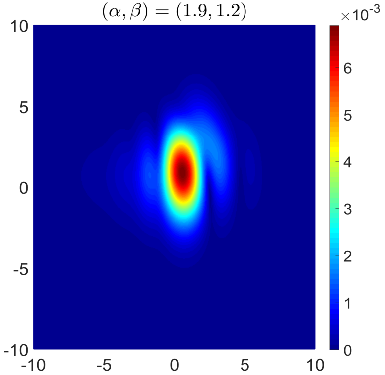

Figure: Comparison of the absolute values of the LBDF2 solution and our low-rank solution at $t = T$ for $(N, M) = (512, 200)$ and $(\alpha, \beta) = (1.9, 1.2)$ for Example X. Left: The LBDF2 solution. Right: The low-rank solution (rank $r = 5$).

Summary

Conclusions

- A numerical integration method based on a dynamical low-rank approximation is proposed to solve 2D FGLEs.
- We conduct an error analysis of the proposed procedure, which is independent of the stiffness and robust with respect to possibly small singular values in the approximation matrix.
- Numerical results show that our method is robust and accurate.

Work(s) in progress

Work(s) in progress/ideas:

- Extension of our method to other problems such as space fractional Schrödinger equations.
- For solving higher-dimensional version of (1), we suggest considering the dynamical tensor approximation[i].
- Design some fast implementations (e.g., a parallel version) of our method.

[i] Ot. Koch and C. Lubich. "Dynamical tensor approximation". In: SIAM J. Matrix Anal. Appl. 31 (2010), pp. 2360–2375.

Our recent work

1) Y.-L. Zhao, A. Ostermann, X.-M. Gu, A low-rank Lie-Trotter splitting approach for nonlinear fractional complex Ginzburg-Landau equations, J. Comput. Phys., 2020, 17 pages. (under review)
2) Y.-L. Zhao, M. Li, A. Ostermann, X.-M. Gu, An efficient second-order energy stable BDF scheme for the space fractional Cahn-Hilliard equation, BIT, 2019, 26 pages. (under revise)
3) Y.-L. Zhao, X.-M. Gu, A. Ostermann, A parallel preconditioning technique for an all-at-once system from subdiffusion equations with variable time steps, J. Comput. Sci., 2019, 22 pages. (under revise)

Questions or comments?

Many thanks for the kind invitation and your attention!